Scheduling Algorithms for Multiprogramming in a Hard Real-time Environment

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Introduction and Background

- The use of computers for
  - Time-critical control and monitoring of industrial processes
- Two scheduling algorithms are studied
  - Both are priority driven and preemptive
    - The processing of any task is interrupted by a request for any higher priority task
  - The first one
    - Fixed priority assignment algorithm
  - The second one
    - Dynamic priority assignment

The Environment

- Five assumptions
  - (A1) The request for all tasks for which hard deadlines exist are periodic
  - (A2) Deadlines consist of run-ability constraints only
    - Each task must be completed before the next request for it occurs
  - (A3) The tasks are independent in that requests for a certain task do not depend on the initiation or the completion of requests for other tasks
  - (A4) Run-time for each task is constant for that task and does not vary with time. Run-time here refers to the time which is taken by a processor to execute the task without interruption
  - (A5) Any nonperiodic tasks in the system are special; they are initialization or fault-recovery routines; they displace periodic tasks while themselves are being run, and do not themselves have hard, critical deadlines

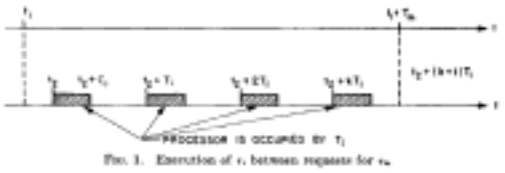
A Fixed Priority Scheduling Algorithm (1)

- **Task and scheduling model**
  - **Deadline of a request**
    - The time of the next request for the same task
  - **Overflow of a request**
    - Unfulfilled request by its deadline
  - **Feasible algorithm**
    - If tasks are scheduled so that no overflow ever occurs
  - **Response time of a request**
    - The time span between the request and the end of the response to that request
  - **Critical instant of a task**
    - An instant at which a request for that task will have the largest response time
  - **Critical time zone for a task**
    - The time interval between a critical instant and the deadline of the corresponding request of the task

A Fixed Priority Scheduling Algorithm (2)

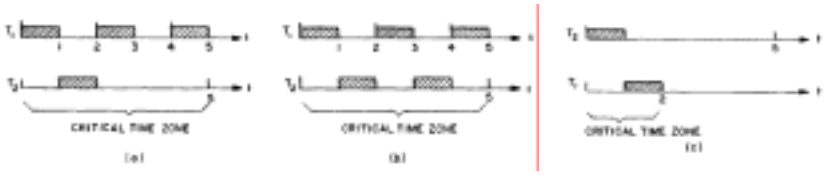
- **Theorem 1. A critical instant for any task occurs whenever the task is requested simultaneously with requests for all higher priority tasks**

- **Proof.**
  - **Advancing the request time t_2 will not speed up the completion of τ_m**



A Fixed Priority Scheduling Algorithm (3)

- Important value of Theorem 1
  - A simple direct calculation can determine whether or not a given priority assignment will yield a feasible scheduling algorithm
- Example: two tasks τ_1 and τ_2
  - $T_1 = 2, C_1 = 1$
  - $T_2 = 5, C_2 = 1$



A Fixed Priority Scheduling Algorithm (4)

- The following inequalities must be met for feasibility
  - If τ_1 is the highest priority task (method 1)
    - $\lfloor T_2/T_1 \rfloor \; C_1 + C_2 \leq T_2$. (1)
  - If τ_2 is the highest priority task (method 2)
    - $C_1 + C_2 \leq T_1$. (2)
  - It follows from (2) that
    - $\lfloor T_2/T_1 \rfloor \; C_1 + \lfloor T_2/T_1 \rfloor \; C_2 \leq \lfloor T_2/T_1 \rfloor \; T_1 \leq T_2$
- Whenever $T_1 \leq T_2$ and τ_2 has the highest priority, Ineq.(2) implies Ineq.(1)
  - Ineq.(1) is a weaker condition
  - If a task set is schedulable by method 2, then it is also schedulable by method 1

A Fixed Priority Scheduling Algorithm (5)

- Reasonable rule of priority assignment
  - Assign priorities according to request rates, independent of run-times
  - Tasks with higher request rates will have higher priorities
- Rate-monotonic priority assignment
  - Optimum in the sense that no other fixed priority assignment rule can schedule a task set which cannot be scheduled by the rate-monotonic priority assignment
- Theorem 2. If a feasible priority assignment exists for some task set, the rate monotonic priority assignment is feasible for that task set
  - Proof. By an inter-change argument

Achievable Processor Utilization (1)

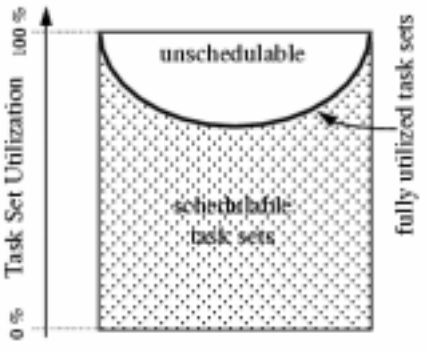
- Utilization factor
  - The fraction of processor time spent in the execution of the task set
    - The utilization factor is equal to one minus the fraction of idle processor time
  - $$U = \sum_{i=1}^{m} (C_i/T_i).$$
  - Utilization factor can be improved by
    - Increasing run-times and decreasing periods
  - However, utilization factor is upper bounded by the requirement of feasibility (meeting deadlines)
  - How large the processor utilization factor can be?

Achievable Processor Utilization (2)

- Full utilization of processor
  - A set of tasks is said to “fully utilize” the processor if the priority assignment is feasible for the set and if an increase in the run-time of any of the tasks in the set will make the priority assignment infeasible
- LUB (least upper bound) of utilization factor
  - The minimum of utilization factor over all sets of tasks that fully utilize the processor
  - For all task sets whose utilization factor is below this bound, there exists a fixed priority assignment which is feasible

Achievable Processor Utilization (3)

- Schedulable Utilization Bound



Achievable Processor Utilization (4)

- **Theorem 3. For a set of two tasks with fixed priority assignment, the least upper bound to the processor utilization factor is** $U = 2(2^{1/2} - 1)$

- **Proof. See Theorem 4**

Achievable Processor Utilization (5)

- **Theorem 4. For a set of m tasks with fixed priority order, and the restriction that the ratio between any two request periods is less than 2, the least upper bound to the processor utilization factor is** $U = m(2^{1/m} - 1)$

- **Proof.**
  - **(1) We first show that the LUB can be found when**
    - $C_1 = T_2 - T_1$, $C_k = T_{k+1} - T_k$, and
    - $C_m = T_m - 2(C_1 + C_2 + ... + C_{m-1}) = 2T_1 - T_m$
  - **(2) We then compute the LUB**

Achievable Processor Utilization (6)

- **We first show that the LUB can be found when**
  - $C_1 = T_2 - T_1, C_k = T_{k+1} - T_k$, **and** $C_m = 2T_1 - T_m$

- **Proof (1). For** $\Delta \geq 0$
  - **(1) Suppose that** $C_1 = T_2 - T_1 + \Delta$ **and the task set S fully utilize**
    - We can construct another set S' such that
    - $C_1^{'} = T_2 - T_1$, $C_2^{'} = C_2 + \Delta$, and $C_k^{'} = C_k$
    - S' fully utilize since $C_1 + C_2 = C_1^{'} + C_2^{'}$
    - $U - U' = \Delta / T_1 - \Delta / T_2 \geq 0$
  - **(2) Suppose that** $C_1 = T_2 - T_1 - \Delta$ **and the task set S fully utilize**
    - We can construct another set S' such that
    - $C_1^{'} = T_2 - T_1$, $C_2^{'} = C_2 - 2\Delta$, and $C_k^{'} = C_k$
    - S' fully utilize
    - $U - U' = -\Delta / T_1 + 2\Delta / T_2 \geq 0$
  - **Similarly, we can show that** $C_k = T_{k+1} - T_k$

Achievable Processor Utilization (7)

- **Illustration**

Case (1) ⇨

Case (2) ⇨

consider this interval

Case for LUB ⇨

Achievable Processor Utilization (8)

- We next compute the LUB

- Using the previous results, we can write

$$U = (T_2 - T_1)/T_1 + (T_3 - T_2)/T_2 + \ldots + (2T_1 - T_m)/T_m$$
$$= T_2/T_1 - 1 + T_3/T_2 - 1 + \ldots + 2T_1/T_m - 1$$
$$= T_2/T_1 + T_3/T_2 + \ldots + 2T_1/T_m - m$$

- Let $R_i = T_{i+1}/T_i$

$$\frac{T_1}{T_m} = \frac{T_1}{T_m}\frac{T_{m-1}\ldots T_3 T_2}{T_{m-1}\ldots T_3 T_2} = \frac{T_{m-1}T_{m-2}\ldots T_3 T_1}{T_m T_{m-1}\ldots T_3 T_2} = 1/R_{m-1}R_{m-2}\ldots R_1$$

$$U = R_1 + R_2 + \ldots + R_i + \ldots + 2/(R_{m-1}\ldots R_1) - m$$

Achievable Processor Utilization (9)

- Given

$$U = R_1 + R_2 + \ldots + R_i + \ldots + 2/(R_{m-1}\ldots R_1) - m$$

- To find the minimum, we solve the following

$$\partial U/\partial R_i = 1 - 2(R_{m-1}\ldots R_1)/(R_{m-1}\ldots R_i\ldots R_1)^2 = 0$$

$$R_i = 2/(R_{m-1}R_{m-2}\ldots R_2 R_1)$$

- This implies $R_1 = R_2 = \ldots = R_{m-1}$, thus $R_i = 2^{1/m}$
- Finally it follows that

$$U = (m-1)2^{\frac{1}{m}} + 2/(2^{\frac{m-1}{m}}) - m = (m-1)2^{\frac{1}{m}} + 2/(2^{1-\frac{1}{m}}) - m$$
$$= (m-1)2^{\frac{1}{m}} + 2^{\frac{1}{m}} - m = m(2^{\frac{1}{m}} - 1)$$

Achievable Processor Utilization (10)

- For m=3, $U = 3(2^{1/3} - 1) \approx 0.78$
- For large m, $U \approx \ln 2$
- The restriction that the largest ratio between request period less than 2 can be removed
- Theorem 5. For a set of m tasks with fixed priority order, the least upper bound to processor utilization is $m(2^{1/m} - 1)$

Achievable Processor Utilization (11)

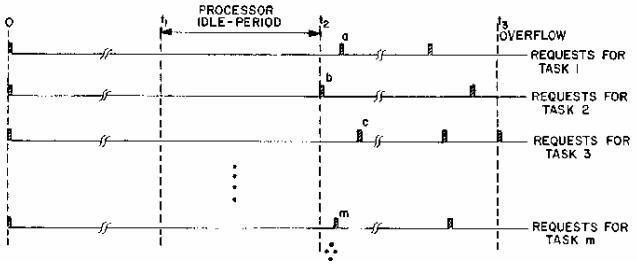
- Outline of proof of Theorem 5
  - The idea is that if a set of tasks fully utilizes the processor and for some i, i < m
    - $T_m/T_i \geq 2$,
  - then we can always construct another set of tasks that will
    - (1) fully utilize the processor,
    - (2) $T_m/T_i < 2$, and
    - (3) the utilization of the new set is less than the original one

The Deadline Driven Scheduling Algorithm (1)

- Priorities are assigned to tasks according to the deadlines of their current requests
  - A task will be assigned the highest priority if the deadline of its current request is the nearest
  - At any instant, the task with the highest priority and yet unfulfilled request will be executed
  - This method of assigning priorities is a dynamic one
- We want to establish a necessary and sufficient condition for the feasibility of the deadline driven scheduling algorithm

The Deadline Driven Scheduling Algorithm (2)

- Theorem 6. When the deadline driven scheduling algorithm is used to schedule a set of tasks on a processor, there is no processor idle time prior to an overflow
- Proof. Shifting a, b, .., c to t1 leads to a contradiction



The Deadline Driven Scheduling Algorithm (3)

- **Theorem 7. For a given set of m tasks, the deadline driven scheduling algorithm is feasible if and only if**

$$U = C_1/T_1 + C_2/T_2 + ...C_m/T_m \le 1$$

- **Proof.**
  - **(1) To show the necessity, compute the total demand of computation time**

$$(T_2T_3 \cdots T_m)C_1 + (T_1T_3 \cdots T_m)C_2 + \cdots + (T_1T_2 \cdots T_{m-1})C_m .$$

$$(T_2T_3 \cdots T_m)C_1 + (T_1T_3 \cdots T_m)C_2 + \cdots + (T_1T_2 \cdots T_{m-1})C_m > T_1T_2 \cdots T_m$$

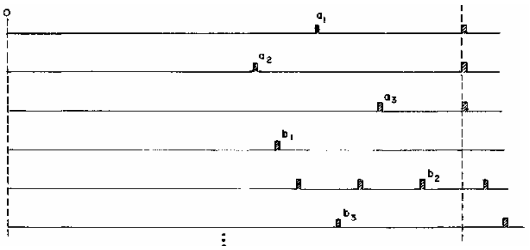
$$(C_1/T_1) + (C_2/T_2) + \cdots + (C_m/T_m) > 1$$

The Deadline Driven Scheduling Algorithm (4)

- **Proof of Theorem 7.**
  - **(2) To show the sufficiency, assume that the algorithm is not feasible and** $U = C_1/T_1 + C_2/T_2 + ...C_m/T_m \le 1$
    - There will be an overflow at $t = T$ in the interval $[0, T_1T_2...T_m]$
    - There is no idle time in the interval [0, T]
    - Let $a_1, a_2, a_3, ..., b_1, b_2, b_3...$ denote the request times immediately before T
    - Let $a_1, a_2, a_3, ...$ are the request times of tasks with deadlines at T
    - Let $b_1, b_2, b_3, ...$ are the request times of tasks with deadlines beyond T

The Deadline Driven Scheduling Algorithm (5)

❒ **Proof of Theorem 7.**



- **Two cases must be considered**
  - Case 1. None of the computation times of requested at $b_1, b_2, b_3, \ldots$ was carried out before T
  - Case 2. Some of the computation times of requested at $b_1, b_2, b_3, \ldots$ was carried out before T

The Deadline Driven Scheduling Algorithm (6)

❒ **Proof of Theorem 7.**

- **Case 1.**
  - The total demand of computation time in the interval [0, T]

$$\lfloor T/T_1 \rfloor C_1 + \lfloor T/T_2 \rfloor C_2 + \cdots + \lfloor T/T_m \rfloor C_m$$

  - Since there is no idle period

$$\lfloor T/T_1 \rfloor C_1 + \lfloor T/T_2 \rfloor C_2 + \cdots \lfloor T/T_m \rfloor C_m > T$$

  - Thus

$$(T/T_1)C_1 + (T/T_2)C_2 + \cdots + (T/T_m)C_m > T$$

$$(C_1/T_1) + (C_2/T_2) + \cdots + (C_m/T_m) > 1$$

  - This is a contradiction

The Deadline Driven Scheduling Algorithm (7)

- **Proof of Theorem 7.**
  - **Case 2.**
    - There must exist exist a point T' such that none of requests at $b_1, b_2, b_3, \ldots$ was carried out in the interval [T', T]
    - In other words, only those requests with deadline at or before T will be executed
    - Note that since some of the requests at $b_1, b_2, b_3, \ldots$ can be executed until T', all those requests initiated before T' with deadlines at or before T have been fulfilled before T'
    - Therefore, the total demand of computation time in the interval [T', T] is less than or equal to

      $$\lfloor (T - T')/T_1 \rfloor C_1 + \lfloor (T - T')/T_2 \rfloor C_2 + \cdots + \lfloor (T - T')/T_m \rfloor C_m$$

    - Since an overflow occurs,

      $$\lfloor (T - T')/T_1 \rfloor C_1 + \lfloor (T - T')/T_2 \rfloor C_2 + \cdots + \lfloor (T - T')/T_m \rfloor C_m > T - T',$$

    - $(C_1/T_1) + (C_2/T_2) + \cdots + (C_m/T_m) > 1$, which is a contradiction